Quasi-orthogonal cocycles and optimal sequences

Dane Flannery, joint with J. A. Armario

August 2, 2019
Cohomology in design theory

Pioneering work by de Launey, Horadam originated the theory of cocyclic pairwise combinatorial designs. A PCD is a square matrix over an ambient ring whose rows (sometimes, also columns) taken pairwise satisfy some fixed constraint, embodied in the orthogonality set chosen for the class of PCDs. Base case of cocyclic development: group development. A $v \times v$ matrix $D$ is group-developed over a group $G$ of order $v$ if $D$ is an image of $G$'s multiplication table, i.e., $D = \left[ \phi(x \cdot y) \right]_{x, y \in G}$, some map $\phi$.

Regular actions on arrays: $D$ is group-developed over $G$ $\iff$ $G$ acts regularly on $D$ (as a group of pairs of permutation matrices): $P_g\left[ \phi(x \cdot y) \right]P_\top g = \left[ \phi(xg \cdot y) \right] = D$.

Induced row and column actions are both regular.

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Group development—purely algebraic definition—gives effective tools from algebra to study PCDs.

But the algebraic nature of a group-developed PCD is restrictive, e.g., group-developed Hadamard matrix must have square order.

Cocyclic development generalizes group development, is less restrictive, and seems to be common for many kinds of PCDs. (Cf. Ito’s Hadamard groups; Craigen’s signed permutation groups.)

\[ G, U \text{ groups, } U \text{ abelian.} \]

\[ Z_2(G,U) := \text{group of all maps } \psi: G \times G \to U \text{ such that } \psi(x,y)\psi(xy,z) = \psi(x,yz)\psi(y,z) \forall x,y,z \in G, \]

called cocycles.

Assume \( \psi \) normalized, i.e., \( \psi(1,1) = 1 \), and display as a cocyclic matrix \( M_\psi = [\psi(g,h)] \)

In group development, cocycles are coboundaries \( \partial \phi \), where \( \partial \phi(x,y) = \phi(x) - 1 - \phi(xy) \) for \( \phi: G \to U \) (the ‘splitting case’).
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Suppose that $|G|$ is divisible by 4. Say $\psi \in Z^2(G, \langle -1 \rangle)$ is orthogonal if $M_\psi$ is Hadamard.
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Note: if $H$ is Hadamard, regular group acting on $\mathcal{E}_H$ is a Hadamard group.

Associativity of multiplication in this central extension of $\langle -1 \rangle$ by $G$ is $(\dagger)$. 

Lemma \(\psi\) is orthogonal $\iff$ no. $+1$s = no. $-1$s in every non-initial row of $M_\psi$. That is, a cocyclic $\langle -1 \rangle$-matrix $H$ is Hadamard iff its row excess $\text{RE}(H) := \sum_{i \geq 2} |\sum_{j \geq 1} h_{i,j}|$ is optimal (least, i.e., zero).
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**Lemma**

\(\psi\) *is orthogonal* \(\iff\) no. +1s = no. -1s in every non-initial row of \(M_\psi\).
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An analog of orthogonal cocycle for orders $\not\equiv 0 \pmod{4}$

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Lemma

Let $M$ be a cocyclic $\langle -1 \rangle$-matrix with indexing group $G$.

(i) Either exactly half the rows of $M$ are even, or all rows are even; thus $\text{RE}(M) \geq 4t$.

(ii) $\text{RE}(M) = 4t$ iff $\text{abs}(MM^\top) = [4tI + 2J_{0}0 0 4tI + 2J_{0}]$ up to row permutation.
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In analogy with definition (characterization) of orthogonal cocycle:

**Definition**

$\psi \in Z^2(G, \langle -1 \rangle)$ is **quasi-orthogonal** if $\text{RE}(M_\psi) = 4t$. 

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\[ \psi \text{ is quasi-orthogonal } \iff |\{g \in G \mid \sum_{h \in G} \psi(g, h) = \pm 2\}| = 2t \text{ and } |\{g \in G \mid \sum_{h \in G} \psi(g, h) = 0\}| = 2t + 1. \]
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**Example.** \(\psi \in Z^2(\text{Sym}(3), \langle -1 \rangle)\) *given by*

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is quasi-orthogonal (three rows sum to 0, two rows sum to 2). But \(\det(M_\psi) = 128\) does not attain the E–W bound \(160\).
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If $|G| = 4t + 2$ and $\psi \in \mathbb{Z}_2(G, \langle -1 \rangle)$ is a coboundary then $\psi$ is not quasi-orthogonal (every row in $M_\psi$ is even, $\text{RE}(M_\psi) \geq 8t + 2$).

Existence of quasi-orthogonal cocycles confirmed by computer $\forall G$ of order $2$ odd $\leq 42$.

Also have infinite families over cyclic groups.

Do quasi-orthogonal cocycles always exist (over every possible group)?

Cf. Ito’s non-existence results for Hadamard groups, & classification by ´O Cath´ain and R¨oder yielding other non-examples.

Also, haven’t yet found $G$ of an allowable order for which there are no quasi-orthogonal cocycles whose matrices attain the E–W bound.
Proving existence of cocyclic PCDs is (computationally) hard. As expected! e.g., Cocyclic Hadamard Conjecture of de Launey and Horadam. Re. existence of quasi-orthogonal cocycles, note

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Theorem

Let $\psi \in \mathbb{Z}_2^G$, where $|G| = 4t + 2$. If either $\psi$ is quasi-orthogonal, or $G$ is abelian or dihedral and $\psi$ is not a coboundary for dihedral $G$, then $M_\psi M_{\psi}^T = M_{\psi}^T M_\psi$. In particular, could have defined quasi-orthogonal cocycle equivalently in terms of optimal column excess. (Recall that $\psi$ is quasi-orthogonal iff $\text{abs}(M_\psi M_{\psi}^T) = I_2 \otimes (4tI_2 + 2J_2).$)

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Sequences and arrays from cocycles

Let \( \phi = (\phi(0), \ldots, \phi(n-1)) \in \\{\pm 1\}^n \) or \( \{\pm 1, \pm i\}^n \).

\[ R_\phi(w) := \sum_{k=0}^{n-1} \phi(k) \phi(k+w), \] periodic autocorrelation of \( \phi \) at shift \( w \).

We have
\[ \max_{0 < w < n} |R_\phi(w)| \geq \begin{cases} \frac{1}{n} & \text{if } n \text{ odd}, \\ 2 & \text{if } n \equiv 2 \mod 4. \end{cases} \]
when \( \phi \) binary, and
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for odd \( n \) when \( \phi \) quaternary.

If \( R_\phi(w) = 0 \) for \( 0 < w < n \) then \( \phi \) is perfect.

Conjecture: perfect sequences over \( m \)th roots of unity of length \( >m^2 \) do not exist.

(For \( m = 4 \) see Arasu, de Launey, Ma, On circulant complex Hadamard matrices, Des. Codes Cryptogr. 25 (2002).)

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A quaternary sequence $\phi$ of length $n$ has optimal autocorrelation ($\phi$ is an OQS) if

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Let $G = \mathbb{Z}_{s_1} \times \cdots \times \mathbb{Z}_{s_r}, s_i > 1$; $s := (s_1, \ldots, s_r)$.

A (binary or quaternary) $s$-array is just a map $\phi: G \to C = \{\pm 1\}$ or $\{\pm 1, \pm i\}$.

A sequence is an $s$-array with $r = 1$.

For a type vector $z = (z_1, \ldots, z_r) \in \{0, 1\}^r$, let $E = \mathbb{Z}(z_1+1)s_1 \times \cdots \times \mathbb{Z}(z_r+1)s_r$, $H = \{ (h_1, \ldots, h_r) \in E | h_i = 0$ if $z_i = 0$, and $h_i = 0$ or $s_i$ if $z_i = 1 \}$, $K = \{ h \in H | h$ has even weight $\}$.

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E &= \mathbb{Z}_{(z_1+1)s_1} \times \cdots \times \mathbb{Z}_{(z_r+1)s_r}, \\
H &= \{(h_1, \ldots, h_r) \in E \mid h_i = 0 \text{ if } z_i = 0, \text{ and } h_i = 0 \text{ or } s_i \text{ if } z_i = 1\}, \\
K &= \{h \in H \mid h \text{ has even weight}\}.
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$H \leq E$ is elementary abelian 2-group
Jedwab investigated generalized perfect binary arrays (Des. Codes Cryptogr. 2, 1992); these are cocyclic (Hughes, European J. Combin. 21, 2000).

Let \( G = \mathbb{Z}_{s_1} \times \cdots \times \mathbb{Z}_{s_r}, s_i > 1; \ s := (s_1, \ldots, s_r). \)

A (binary or quaternary) \( s \)-array is just a map \( \phi : G \to C = \{\pm 1\} \) or \( \{\pm 1, \pm i\} \). A sequence is an \( s \)-array with \( r = 1 \).

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Now let $|G| \equiv 2 \mod 4$, say $s_1/2, s_2, \ldots, s_r$ are odd.

A generalized optimal binary array of type $z$, GOBA($s$), is a binary $s$-array $\phi$ such that

- $R_{\phi'}(x) \in \{0, \pm 2 |H|\} \forall x \in E \setminus H$
- $|\{x \in E | R_{\phi'}(x) = 0\}| = |E|/2$ if $z_1 = 1$.

A generalized optimal binary sequence (GOBS) has $r = z_1 = 1$.

**Theorem**

Let $\phi$ be a binary sequence of length $2^m$, $m > 1$ odd. Then $\phi$ is a GOBS($2^m$) ⇔ there is a GOBA($2$, $m$)$\phi$ of type $(1, 0)$ ⇔ there is a quasi-orthogonal cocycle $\psi \in Z_2(Z_2^m, \langle -1 \rangle)$.

**Proof:**

Use isomorphism $Z_2 \times Z_m \cong Z_2^m$ to pass between $\phi$ and $\phi'$, deploy signs suitably; $\psi = f_{z} \partial \phi$, where $f_{z} \not\in B_2(Z_2^m, \langle -1 \rangle)$. 

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Quasi-orthogonal cocycles and optimal sequences
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Summary of the variations for cocyclic arrays

Can vary: (i) entries (binary or quaternary); (ii) length ($\equiv 0$ or $\equiv 2 \mod 4$), orthogonal or quasi-orthogonal; (iii) split or not; (iv) dimension ($1$ or $>1$).

For binary arrays:

- Orthogonal cocycle ($n \equiv 0 \mod 4$)
- Quasi-orthogonal cocycle ($n \equiv 2 \mod 4$)

1-d non-split: GPBS
1-d non-split: GOBS ($>1$)
1-d split: PS
1-d split: OBS ($>1$)
1-d split: PBA ($>1$)
1-d split: OBA ($>1$)

Exercise: fill in all known existence results in each case.

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Quasi-orthogonal cocycles and optimal sequences
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Optimal quaternary sequences of odd length

Looking for quasi-orthogonal cocycles over the most basic kind of indexing group, cyclic (of order \(2^{\text{odd}}\); say \(m\) odd).

Theorem

The following are equivalent:

- OQS of odd length \(m\);
- GOBS \((2^m)\);
- GOBA \((2^m, m)\) of type \((1, 0)\);
- quasi-orthogonal cocycles over \(\mathbb{Z}_{2^m}\).

Example.

\[
\begin{bmatrix}
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\end{bmatrix}
\]

is a GOBA\((2, 3)\),

\[
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is a GOBA\((2^2, 5)\), both of type \((1, 0)\).

The corresponding OQS are \((1, i, 1\)\

\(R^* = (3, 1, 1)\) and \(R^* = (5, 1, 1, 1)\).

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Theorem

The following are equivalent: OQS of odd length $m$; GOBS$(2m)$; GOBA$(2, m)$ of type $(1, 0)$; quasi-orthogonal cocycles over $\mathbb{Z}_{2m}$.

Example. $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ is a GOBA$(2, 3)$, $\begin{bmatrix} 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ is a GOBA$(2, 5)$, both of type $(1, 0)$.

The corresponding OQS are $(1, i, 1)$, $(1, -1, 1, 1, 1)$, with $R_* = (3, 1, 1)$ and $R_* = (5, 1, 1, 1, 1)$.

Their GOBS are $(1, 1, -1, -1, -1, 1)$ and $(1, -1, -1, -1, 1, 1, 1, -1, 1, 1)$. 

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- $m = (p^a + 1)/2$, $p$ prime (C-sequences of Schotten).