The Ryser Design Conjecture
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Sharad S. Sane

Chennai Mathematical Institute
Sipcot IT Park, Siruseri
Chennai 603103
ssane@cmi.ac.in, sharadsane@gmail.com

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In the words of Gian Carlo Rota (Preface: Studies in Combinatorics published by the MAA):

Block Designs are generally acknowledged to be the most complex structures that can be defined from scratch in a few lines. Progress in understanding and classification has been slow and has proceeded by leaps and bounds, one ray of sunlight being followed by years of darkness. . . . The subject has been made even more mysterious, a battleground of number theory, projective geometry and plain cleverness. This is probably the most difficult combinatorics going on today . . .
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Section 1

Symmetric Designs
Many constructions of symmetric designs are known. The existence question for Symmetric designs is the question of constructing a \((0,1)\)-matrix satisfying the matrix equations given above. Algebraic number theory has been employed in order to answer this existence question and the relevant seminal result is called the Bruck-Ryser-Chowla theorem. Unfortunately, it works only in one direction. That is, it provides us with only a necessary condition which, may not be sufficient. For example, it is not known whether there is a projective plane of order twelve but it is known, thanks to the Bruck-Ryser-Chowla theorem that
If $q$ is the order of a projective plane such that $q \equiv 1, 2 \ (mod \ 4)$ then $q$ is a sum of two integer squares. In particular, there are no projective planes of orders $q$ such that $q \equiv 6 \ (mod \ 8)$.
The non-existent plane of order ten

An extensive search for almost 200 hours on the fastest CRAY computer available then proved in the late 1980s that there is no projective plane of order ten.
On all the Known Constructions

- When $\lambda = 1$, we have a projective plane of order $q$ with parameters $(q^2 + q + 1, q + 1, 1)$. These exist for every prime power $q$. No other examples are known.

- When $\lambda = 2$, we have a biplane with parameters $\left(\binom{k}{2} + 1, k, 2\right)$. These are known to exist for the following values of $k$: $3, 4, 5, 6, 9, 11, 13$. No other examples are known.

- When $\lambda = 3$, all the known examples have $k$ bounded by 15.

- When $\lambda \geq 4$, all the known examples have $k$ bounded by $\lambda^2 + \lambda$. 
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On all the Known Constructions
Conjectures

The known situation led to the following informal conjecture attributed to M. Hall.

M. Hall’s conjecture: \( \forall \lambda \geq 2 \), there exist only finitely many symmetric \((v, k, \lambda)\).

Stronger: \( \forall \lambda \geq 4 \), \( k \) satisfies \( k \leq \lambda^2 + \lambda \).
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Section 2

Quasi-Symmetric Designs
Take a design $D$, not necessarily symmetric. An integer $x$ is called a block intersection number of $D$ if we have two blocks $X$ and $Y$ the cardinality of whose intersection is $x$. Which numbers occur as block intersection numbers of a design? Thanks to the proof of Fisher’s inequality, we see that $D$ has exactly one block intersection number iff it is a symmetric design.
A design with two block intersection numbers $x$ and $y$ (with $x < y$ by convention) is called a Quasi-symmetric design.

Reasons for studying quasi-symmetric designs are many. A mundane and practical reason is that symmetric designs are more difficult to study (this is not completely true but sometimes believed to be so). On a more serious level quasi-symmetric design allows one to construct its block graph which in most cases of interest can be shown to be strongly regular. Finally quasi-symmetric designs are connected with combinatorial configurations arising out of finite simple groups.
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Let $D$ be a point-block incidence structure constructed from $AG(n, q)$ an affine geometry of dimension $n$ over a field with $q$ elements by taking as points the points of the geometry and as blocks all the affine hyperplanes (where $n \geq 2$). Parameters of $D$ as a quasi-symmetric design are:

\[ v = q^n, \quad k = q^{n-1} \]

\[ \lambda = q^{n-2}, \quad x = 0, \quad y = q^{n-2} \]
Let $D$ be a point-block incidence structure constructed from $PG(n, q)$ a projective geometry of dimension $n$ over a field with $q$ elements by taking as points the points of the geometry and as blocks the subspaces of codimension two (where $n \geq 3$). Parameters of $D$ as a quasi-symmetric design are:

\[ v = \frac{q^{n+1} - 1}{q - 1}, \quad k = \frac{q^{n-1} - 1}{q - 1} \]

\[ \lambda = \frac{q^{n-1} - 1}{q - 1}, \quad x = \frac{q^{n-3} - 1}{q - 1}, \quad y = \frac{q^{n-2} - 1}{q - 1} \]
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$$\begin{align*}
v &= \frac{q^{n+1} - 1}{q - 1}, \\
k &= \frac{q^{n-1} - 1}{q - 1}, \\
\lambda &= \frac{q^{n-1} - 1}{q - 1}, \\
x &= \frac{q^{n-3} - 1}{q - 1}, \\
y &= \frac{q^{n-2} - 1}{q - 1}
\end{align*}$$
There are other classes of examples particularly the affine geometries (where $x = 0$). There is also a classical object called the Witt design on 23 points which is associated with the Mathieu group $M_{23}$ on 23 letters.
Section 3

Linear algebra of Quasi-symmetric designs
Define the block graph $\Gamma$ of a quasi-symmetric design $D$ by taking as vertices of $\Gamma$ the blocks of $D$. Make two vertices adjacent iff the corresponding blocks intersect in $x$ points. Let $N$ denote the incidence matrix of $D$ and $A$, the adjacency matrix of $\Gamma$. Recall that we have already established the following matrix equations:

$$NN^t = (r - \lambda)I + \lambda J, \quad N^t J = kJ, \quad NJ = rJ$$ (1)
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Here, $D$ is quasi-symmetric and hence the following matrix equation connects $N^tN$ and $A$:

$$N^tN = kI + xA + y(J - I - A)$$

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Since $N^tN$ commutes with $J$ it follows that $A$ also commutes with $J$ which is just the same thing as saying that $\Gamma$ is a regular graph. If we can now show that besides the degree of regularity, $A$ has exactly two other eigenvalues, then $\Gamma$ must be a strongly regular graph. But $NN^t$ and $N^tN$ have the set of non-zero eigenvalues. So, $N^tN$ has eigenvalues $rk, r - \lambda$ and 0. The eigenvalue $rk$ is a simple eigenvalue of $N^tN$ that corresponds to the largest simple eigenvalue of $A$ which is also the degree of regularity of $\Gamma$. Hence $A$ has exactly two other eigenvalues proving that the block graph is strongly regular.
The fact that under the mild condition that the block graph and its complement be connected is sufficient to frame the study of quasi symmetric designs in terms of strongly regular graphs has some nice consequences. If one treats a symmetric design as a degenerate (or limiting) case of a quasi symmetric design, then a coauthored result of mine shows that for a fixed pair \((x, y) = (0, y)\) of block intersection numbers, if we let \(k \to \infty\), then the resulting structure has to turn out to be a symmetric design showing in some sense that:
A formulation of M. Hall conjecture for the general class of quasi symmetric designs with $x = 0$ is not considerably harder than Hall conjecture for symmetric designs and thus the real difficulty lies in the symmetric case.

A large number of results proved in the last thirty years are around the theme outlines on the previous slides. Among these is the classification result for quasi-symmetric 3-designs in terms of a certain Diophantine equation and its use in studying the 4-design problem that was solved by Ray-Chaudhuri and Wilson.
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Section 4

Ryser design
In a combinatorially dual set up consider a configuration with equally many points and blocks say $v$ with the property that all blocks are proper subsets of the point set and have sizes strictly larger than $\lambda$ such that every two blocks intersect in $\lambda$ points. An example of this situation is of course a symmetric design that we encountered earlier. In fact, if we make a Further assumption that all blocks have the same size $k$ or dually every point is replicated the same number of times, then the configuration must indeed be a symmetric design as shown by the following calculation. for a fixed block $X$ counting the set of all the flags $(x, Y)$ with $x \in X \cap Y$ gives us:

$$k(r - 1) = \lambda(v - 1)$$
Are there examples of incidence structures as we defined that are other than the symmetric designs? This will require that we have at least two replication numbers. Woodall and Ryser constructed families of such configurations from symmetric designs themselves as the basic ingredients.

Let $D$ be a symmetric $(v, k, \lambda)$-design. Fix a block $X$ of $D$. Construct a new configuration $D'$ whose blocks are $X$ and $Y'$ for every $Y \neq X$:

$$Y' = Y \Delta X = (Y - X) \cup (X - Y)$$

This gives us an incidence structure with $v' = v$ and $\lambda' = k - \lambda$. 
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This gives us an incidence structure with $v' = v$ and $\lambda' = k - \lambda$.
Now we have two block sizes: $k$ and $2(k - \lambda)$ as also two replication numbers. Notice that except in the degenerate case when $k = 2\lambda$ the resulting incidence structure is not a symmetric design. This prompts the following definition.

An incidence structure (with $v$ points and with every two blocks intersecting in $\lambda$ points) is called a **Ryser design** if it has at least two replication numbers (and is therefore not a symmetric design).

**We also define:**

A Ryser design is said to be a type-1 Ryser design if it is obtained from a symmetric design precisely in the manner we just described.
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We also define:

A Ryser design is said to be a type-1 Ryser design if it is obtained from a symmetric design precisely in the manner we just described.
The Conjecture: Every Ryser design is a type-1 Ryser design.
Section 5

The Fundamental Result and its Proof
Ryser strongly believed in the validity of his conjecture. The conjecture has also been verified for values of $\lambda$ at least up to 100. To the end of proving the conjecture, the most fundamental result was proved by Ryser. This result states that any Ryser design has exactly two replication numbers. Though all the known proofs of this result are linear algebraic in nature, the one we present here is more modern and is essentially based on some change of basis arguments in a finite dimensional vector space.

**Theorem:** Let $D$ be a Ryser design. Then $D$ has precisely two replication numbers $r_1 \neq r_2$ such that $r_1 + r_2 = v + 1$. 
Consider the vector space $V$ of all the linear functions $g$ in $n$ variables $x_1, x_2, \ldots, x_n$ where $g$ has the form:

$$g(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} a_i x_i + b$$

where $a_i$s as also $b$ are constants that come from some (fixed) field. Then $V$ is clearly an $(n + 1)$-dimensional vector space with a natural basis that consists of the $n$ functions $\{x_j\}$ and the constant function 1. Note that we are already given a Ryser design $D$ over a set $S$. Label the points in $S$ by $1, 2, \ldots, n$. Let $g$ be given as above and let $T \subset S$. Then $g(T) = \sum_{i \in T} a_i + b$. We thus express $T$ in its binary form and then $g(T)$ is nothing but $g$ evaluated at $x_i = 1$ for $i \in T$ and $x_i = 0$ for $i \notin T$.

The proof then essentially consists of finding different suitable bases and a clever use of a change of basis formula.
Idea of the proof

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The proof then essentially consists of finding different suitable bases and a clever use of a change of basis formula.
Some details are as follows. For a block $X$ of $D$ define the (scaled characteristic) function $f_X$ in the following manner.

$$f_X(x_1, x_2, \ldots, x_v) = \left( \sum_{i \in X} x_i \right) - \lambda$$  \hspace{1cm} (2)$$

The defining properties of a Ryser design then readily give:

$$f_X(X) = |X| - \lambda$$

$$f_X(Y) = 0 \text{ if } Y \neq X$$
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These properties readily tell us that $f_X$s are linearly independent. We further claim that the constant function 1 is linearly independent of the $f_X$s:
Suppose $1 = \sum_{X} a_{X} f_{X}$. Then applying the right side to a block $Y$ gives us $a_{Y}(|Y| - \lambda) = 1$ and hence

$$a_{Y} = \frac{1}{|Y| - \lambda}$$

We thus obtain:

$$1 = \sum_{X} \frac{f_{X}}{|X| - \lambda}.$$

But then application of the right side to the empty set $\emptyset$ gives

$$\frac{-\lambda}{|\emptyset| - \lambda}$$

on each summand of the right side showing that the right side is negative while the left is clearly positive which is a contradiction.
Fix $i \in S = \{1, 2, \ldots, \nu\}$. Then we have:

$$x_i = \sum_X a_X f_X + \beta$$

(3)

where the scalars are to be (uniquely) determined. Apply both sides to a block $Y$ to obtain the right side equal to $a_Y(|Y| - \lambda) + \beta$ while the left side equals 1 or 0 depending on whether or not $i \in Y$. So:

$$a_X = \begin{cases} 
\frac{1 - \beta}{|X| - \lambda} & \text{if } i \in X \\
-\frac{\beta}{|X| - \lambda} & \text{if } i \notin X 
\end{cases}$$
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a_X = \begin{cases} 
1 - \beta & \text{if } i \in X \\
- \frac{\beta}{|X| - \lambda} & \text{if } i \notin X
\end{cases}
\]
Hence the original equation now reads:

\[ x_i = (1 - \beta) \sum_{i \in X} \frac{f_X}{|X| - \lambda} - \beta \sum_{i \notin X} \frac{f_X}{|X| - \lambda} + \beta \] (4)

Note again that \( \beta \) is a constant that depends only on \( i \). Now apply both sides of (5) to \( T = \emptyset \) and \( T = \{i\} \) respectively to get:

\[ 0 = (1 - \beta)(-\lambda) \sum_{i \in X} \frac{1}{|X| - \lambda} - \beta(-\lambda) \sum_{i \notin X} \frac{1}{|X| - \lambda} + \beta \] (5)

and

\[ 1 = (1 - \beta)(1 - \lambda) \sum_{i \in X} \frac{1}{|X| - \lambda} - \beta(-\lambda) \sum_{i \notin X} \frac{1}{|X| - \lambda} + \beta \] (6)
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On subtraction, we get:

\[ 1 = (1 - \beta) \sum_{i \in X} \frac{1}{|X| - \lambda} \tag{7} \]

Indeed then \( \beta \neq 1 \) and we also have:

\[ \sum_{i \in X} \frac{1}{|X| - \lambda} = \frac{1}{1 - \beta} \tag{8} \]
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substitution of this in (6) gives:

\[
\sum_{i \notin X} \frac{1}{|X| - \lambda} = \frac{1}{\beta} - \frac{1}{\lambda}
\]
Equations (9) and (10) finally yield:

\[
\frac{1}{\lambda} + \sum_{X} \frac{1}{|X| - \lambda} = \frac{1}{\beta(1 - \beta)} \tag{10}
\]

Observe that the left side is a constant that depends only on the incidence structure $D$ showing that the product $\beta(1 - \beta)$ is a constant. It is equally easy to see from all the equations that $0 < \beta < 1$. Hence we have only two possibilities for $\beta$ in equation (4): either it equals $\beta$ or $1 - \beta$.

Applying equation (7) to the set $S$, we get:

\[
1 = (1 - \beta)r_i - \beta(n - r_i) + \beta \tag{11}
\]

and hence $r_i = \beta(v - 1) + 1$. Or, if $\beta$ is replaced by $1 - \beta$, then $r_i = (1 - \beta)(v - 1) + 1$.

Exactly two replication numbers whose sum is $v + 1$. 
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and hence \( r_i = \beta(v - 1) + 1 \). Or, if \( \beta \) is replaced by \( 1 - \beta \), then \( r_i = (1 - \beta)(v - 1) + 1 \).

Exactly two replication numbers whose sum is \( v + 1 \).
Equations (9) and (10) finally yield:

\[
\frac{1}{\lambda} + \sum_{X} \frac{1}{|X|} - \lambda = \frac{1}{\beta(1 - \beta)} \tag{10}
\]

Observe that the left side is a constant that depends only on the incidence structure D showing that the product \( \beta(1 - \beta) \) is a constant. It is equally easy to see from all the equations that \( 0 < \beta < 1 \). Hence we have only two possibilities for \( \beta \) in equation (4): either it equals \( \beta \) or \( 1 - \beta \).

Applying equation (7) to the set \( S \), we get:

\[
1 = (1 - \beta)r_i - \beta(n - r_i) + \beta \tag{11}
\]

and hence \( r_i = \beta(v - 1) + 1 \). Or, if \( \beta \) is replaced by \( 1 - \beta \), then \( r_i = (1 - \beta)(v - 1) + 1 \).

Exactly two replication numbers whose sum is \( v + 1 \).
Section 6

Discussing known results
Ryser Design conjecture is true if $\lambda = p$ or $\lambda = 2p$ where $p$ is a prime number.

Ryser design conjecture is true if $v = \alpha p + 1$ where $\alpha = 1, 2, \cdots$ is a small natural number.

Some partial results are known on the validity of the conjecture under some extra assumptions, that includes the strong assertion that we have only two block sizes.
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Some partial results are known on the validity of the conjecture under some extra assumptions, that includes the strong assertion that we have only two block sizes.
Tushar Parulekar and S: Ryser design conjecture is true if \( v = 2^n + 1 \).
One of the important points that does not seem to have been sufficiently exploited in this area of research is the following.

Let the point replication numbers of a Ryser design $D$ be $r_1 > r_2$. Then any block complementation of $D$ (which does not result in a symmetric design) has the same replication numbers $r_1$ and $r_2$.

If $e_i$ denotes the number of points with replication number $r_i$ (with $i = 1, 2$), then block complementation does change $e_i$s (subject to $e_1 + e_2 = v$, in general, but must retain the same values $r_i$s.

Tushar Parulekar and S: Let $X$ be any block. Then $r_1 > |X| > r_2$. 
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Tushar Parulekar and S: Let $X$ be any block. Then $r_1 > |X| > r_2$. 
Let \( g \) denote the g.c.d. of \( r_1 \) and \( r_2 \) and let \( c \) and \( d \) respectively denote \( \frac{r_1}{g} \) and \( \frac{r_2}{g} \). Then all the block sizes are in an arithmetic progression with a common difference \( c - d \). This leads to the following. Call a block \( X \) small, average or large depending on \( |X| < 2\lambda \), \( |X| = 2\lambda \) and \( |X| > 2\lambda \) respectively.

(a) If \( \mathbf{D} \) has all the blocks that are large or average, then \( \mathbf{D} \) is of type 1.
(b) If \( \mathbf{D} \) has all the blocks that are small or average, then \( \mathbf{D} \) is of type 1.
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Section 7

Conclusions
Set theoretic Symmetric difference plays an important role in the entire discussion on Type 1 Ryser designs. This is an interesting but equally complex set theoretic operation. In particular we have the following:

Let \( \{ A_1, A_2, \cdots, A_n \} \) be a family of subsets of a universal set \( S \). Then

\[
x \in A_1 \Delta A_2 \Delta \cdots \Delta A_n
\]

holds exactly when \( x \) is in an odd number of \( A_i \)'s.

A consequence of this is the following. Define two Ryser designs \( D \) and \( E \) to be equivalent (written \( D \sim E \)) if one can be obtained from the other by block complementation.

**Theorem:** \( \sim \) is an equivalence relation on the set of all Ryser designs.
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**Theorem:** \( \sim \) is an equivalence relation on the set of all Ryser designs.
Theorem: Let $U$ denote an equivalence class of Ryser designs. Then the following are equivalent.

(a) One design in $U$ is a symmetric design.
(b) One design in $U$ is a Ryser design of type 1.
(c) All designs in $U$ are Ryser designs of type 1.
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THANKS VERY MUCH FOR BEING PATIENT WITH ME.