Satellite Motion in a Yukawa Potential with Drag

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Abstract

Many studies have theorized various modified gravitational models that predict the existence of non-Newtonian corrections to the classical gravitational potential. In this presentation, we discuss the presence of a Yukawa-type gravitational potential and apply it to the case of a satellite under the influence of an atmospheric drag force that varies with the square of the velocity. Using an exponential atmosphere that varies with the orbital altitude of the satellite, we primarily examine a circular orbit scenario by deriving an expression for change in satellite radial distance as a function of drag force parameters due to Yukawa and Newtonian effects, and also by simulating this expression with various numerical possibilities to obtain numerical and graphical results. It is found that Yukawa potential is present in small quantities relative to the Newtonian potential for satellite dynamics and can be further applied in other multi-body systems that may include other celestial objects, such as stars, planets, pulsars and galaxies.

Introduction

The theory of general relativity is well known and proven in various experiments to be one of the building blocks leading to the understanding of physics today. When considering current theories in gravitation, derivations from Newtonian gravity is needed. Haranas and Mioc (2009) have studied the potential of a satellite in motion undergoing a potential such as Maneff potential. This leads to unexpected results which yields a more realistic astronomical result, in comparison to the classical Newtonian model. Many observations of this nature are seen to have a disagreement with the inverse square force law (Haranas et al. 2010). When considering realistic celestial systems, general relativity is inapt in predicting accurate predictions since most of these celestial systems cannot be considered as an isolated system. Forces expect to be significant for pairs of celestial bodies that lie in a mutual distance greater than 1010m (Haranas et al. 2011). More recently, there have be numerous attempts to alter classical gravity to compensate for the long astronomical, astrophysical and cosmological scale which realistic celestial mechanics require. Yukawa potential often suggests a change to the typical Newtonian potential. Yukawa-type additional accelerations have been modeled for various astronomical situations such as solar system effects, astrophysical and cosmological scenarios. Yukawa potential between two bodies is given as:

$$V = -\frac{GMm}{r}\left(1+\alpha e^{-\frac{r}{\lambda}}\right) \quad (1)$$

Where:
G: Gravitational constant
M: primary body of mass
m: secondary body of mass
r: distance between two bodies
λ: range of this interaction
$\alpha = \frac{kK}{GMm}$, where k and K are coupling constants of the new force to the bodies relative to the gravitational one

Our focus here is to explore the two body Yukawa effects on a satellite's motion (Yukawa drag). We find the change in total energy, more specifically we expand on the change in radial distance with respect to different potential aspects such as elliptic height and density

The Yukawa potential

In the 1930's a Japanese theoretical physicist named Hideki Yukawa proposed a new potential of the form, Eq. (1) (Yukawa 1935). When considering a two-body system, the correcting Newtonian correction described in terms of the modified potential can be seen in Eq. (1). The corresponding force can be written as (Fischbach et al.):

$$F = -\frac{GMm}{r^2}\left[1+\alpha e^{-\frac{r}{\lambda}}\right]\hat{r} \quad (2)$$

Where:
$\hat{r}$ is the unit vector in the radial direction

Satellite Motion with Drag

In addition to Yukawa potential, we also take into account the effects of atmospheric drag on the satellite. The relevance of atmospheric drag increases with decreasing altitude in Earth's orbit. Low Earth orbit (LEO) satellites have finite life spans due to the effects of atmospheric drag and must travel at very high speeds through the low-density layers of this medium for years to maintain their orbits. However, the effects of drag are cumulative and will eventually inhibit control over the satellite's altitude reduction. The acceleration due to drag for an LEO satellite is given by (Vallado and McClain, 2007):

$$a_d = -\frac{1}{2}\frac{C_d A_s}{m_s}\rho(r)v_{s_{rel}}^2 \quad (3)$$

Where:
C_d: coefficient of drag
A_s: cross-sectional area of the satellite
$\rho(r)$: atmospheric density function
m_s: mass of satellite
$v_{s_{rel}} = v_s - (\omega_A \times r)$: relative velocity with respect to a rotating atmosphere
ω_A : represent the angular rotation velocity vector of the atmosphere
r: radial vector of the satellite.

The atmospheric density function is assumed as follows (Vallado and McClain, 2007):

$$\rho(h) = \rho_0 e^{-\frac{(h_{el}-h_0)}{H}} \quad (4)$$

where ρ_0 and h_0 are corresponding reference densities and heights, h_{el} is the actual altitude above the ellipsoid and H is the atmospheric scale height. For simplicity, we will disregard the rotational velocity of Earth's atmosphere. Also, the orientation of the satellite's orbit is assumed to be circular. This is because elliptical-orbiting satellites experience the most drag at the perigee, where the satellite encounters its deepest stage inside Earth's atmospheric layers. Drag here causes a decrease in the satellite's velocity, leading to a gradual decrease in its apogee height over time, and thus, equality between the apogee and perigee height. In this near circular motion, air drag acts on the satellite in the tangential direction.

Calculations

Assuming that the satellite follows circular motion, we can calculate the circular orbital velocity to be:

$$v_{tot}^2 = v_N^2 + v_{Yuk}^2 = \frac{GM}{r}\left[1+\alpha\left(1+\frac{r}{\lambda}\right)e^{-\frac{r}{\lambda}}\right] \quad (5)$$

where v_N and v_{Yuk} are the Newtonian and the Yukawa components for the total orbital circular velocity. Air drag changes the energy of the orbit, since the drag acceleration a_d does work on the satellite at $a_d \cdot V = -va_d$ where the negative sign is due to the fact that drag opposes the velocity. Next, we obtain the total energy per unit mass using the relation:

$$E_{tot} = \frac{1}{2}v_{tot}^2 - V_{tot} \quad (6)$$

Eliminating v_{tot}^2 using Eq. (5), and also using Eq.(1) we obtain the total energy per unit mass:

$$E_{tot} = -\frac{GM}{2r}\left[1+\alpha\left(1-\frac{r}{\lambda}\right)e^{-\frac{r}{\lambda}}\right] \quad (7)$$

To proceed with the calculation of the change in radial distance, let us calculate the change of energy in one revolution. Integrating for a full revolution, we write that:

$$\Delta E = -\int_0^{2\pi}\frac{A_s C_d \rho_0 GM}{2r}e^{-\frac{(h_{el}-h_0)}{H}}r\left[1+\alpha\left(1+\frac{r}{\lambda}\right)e^{-\frac{r}{\lambda}}\right]d\theta \quad (8)$$

Furthermore, assuming a small radial change per revolution, and also a constant density over one satellite period, the energy loss becomes:

$$\Delta E = -\pi GMA_s C_d \rho_0 e^{-\frac{(h_{el}-h_0)}{H}}\left[1+\alpha\left(1+\frac{r}{\lambda}\right)e^{-\frac{r}{\lambda}}\right] \quad (9)$$

Taking into account that Yukawa potential modifies the gravitational constant G into a radial distance dependent $G(r)$ that is given by:

$$G(r) = G_\infty\left[1+\alpha\left(1+\frac{r}{\lambda}\right)e^{-\frac{r}{\lambda}}\right] \quad (10)$$

Thus, we can write Eq. (9) as follows:

$$\Delta E_{tot} = -\pi G(r)MA_s C_d \rho_0 e^{-\frac{(h_{el}-h_0)}{H}} = -\pi\mu(r)A_s C_d \rho_0 e^{-\frac{(h_{el}-h_0)}{H}} \quad (11)$$

Where $\mu(r) = G(r)M$. Next, writing the change in the total energy of the satellite we obtain:

$$\Delta E_{tot} = m_s\Delta\left(-\frac{GM}{2r}\left[1+\alpha\left(1-\frac{r}{\lambda}\right)e^{-\frac{r}{\lambda}}\right]\right) = \frac{GMm_s}{2r^2}\left[1+\alpha\left(1+\frac{r}{\lambda}-\frac{r^2}{\lambda^2}\right)e^{-\frac{r}{\lambda}}\right]\Delta r \quad (12)$$

Equating Eq. (11) and Eq. (12) and solving for Δr, we find the total change in the satellite radial distance:

$$\Delta r_{tot} = \Delta r_N + \Delta r_{Yuk} = \frac{-2\pi A_s C_d}{m_s}\rho_0 e^{-\frac{(h_{el}-h_0)}{H}}r^2\lambda\left[\frac{\lambda e^{\frac{r}{\lambda}}+\alpha(r+\lambda)}{\lambda^2 e^{\frac{r}{\lambda}}+\alpha(\lambda^2+\lambda r-r^2)}\right] \quad (13)$$

where Δr_N and Δr_{Yuk} are the change in radial distance due to the contribution of the Newtonian and Yukawa potential, respectively. Δr_N is found to be (Kirk et Al, 2012):

$$\Delta r_N = -\frac{2\pi A_s C_d}{m_s}\rho_0 e^{-\frac{(h_{el}-h_0)}{H}}r^2 \quad (14)$$

Finally, using Eq. (14), we are able to find the change in radial distance due to the contribution of the Yukawa potential only:

$$\Delta r_{Yuk} = \frac{2\pi A_s C_d}{m_s}\rho_0 e^{-\frac{(h_{el}-h_0)}{H}}r^2\left(1-\lambda\left[\frac{\lambda e^{-\frac{r}{\lambda}}+\alpha(r+\lambda)}{\lambda^2 e^{\frac{r}{\lambda}}+\alpha(\lambda^2+\lambda r-r^2)}\right]\right) \quad (15)$$

Numerical Results

For further analysis of the previously derived equations, the change in radial distance (Δr) between the center of mass of two orbiting bodies is explored using the Earth as the main body and an arbitrary satellite as the secondary body. Assuming a circular orbit (e=0) at a radial distance from the center of the Earth where r = 7125 km; yields the orbital amplitude of the satellite to be 747km. Using Eq. (4) and the scale height H=88.667km, the density with respect to height becomes: $\rho \approx 2.123 \times 10^{-14} kg/m^3$ (Vallado and McClain. 2007). Table 1.0 shows the change of radial distance of a satellite moving in a Yukawa gravitational field in circular orbit around the Earth. It can be seen that the effect of the Yukawa potential is in the magnitude 10^{-10}m.

In Figure 1.0, we plot the total radial distance change Δr as a function of orbital reference altitude (h_{el}) between the Earth and a satellite with a surface area of 3m² for multiple aerodynamic coefficients (C_d) ranging from 2.0-2.4 with a mass of 900kg. As seen in Figure 1.0, as the orbital reference increases, the change in radial distance increases. It can be noted that the shapes of the multiple functions are similar when altering the aerodynamic drag coefficient.

In Figure 2.0 we plot the radial distance change Δr as a function of atmospheric density in the reference altitude h = 700-800km for satellites with a mass of 1000kg. Various plots were made to depict multiple surface areas (A_s). As seen in Figure 2.0, there is a decrease in the change in radial distance as the atmospheric density increases which is expected. Furthermore, the increase of surface area also decreases the change in radial distance.

Satellite mass m (kg)	Satellite radial distance r (km)	Aerodynamic drag coefficient C_d	Δr_N (m/rev)	Δr_{yuk} (m/rev)	Δr_{tot} (m/rev)
900	7125.34	2.0	-0.0150785614029	4.151×10^{-10}	-0.0150785613596
		2.1	-0.0158324894731	4.357×10^{-10}	-0.0158324894276
		2.2	-0.0165864175432	4.565×10^{-10}	-0.0165864174954
		2.3	-0.0173403456134	4.772×10^{-10}	-0.0173403455635
		2.4	-0.0180942736836	4.980×10^{-10}	-0.0180942736315
900	6728.137	2.0	-0.0134442720285	3.721×10^{-10}	-0.0134442719919
		2.1	-0.0141164856299	3.907×10^{-10}	-0.0141164855915
		2.2	-0.0147886992313	4.094×10^{-10}	-0.0147886991911
		2.3	-0.0154609128327	4.280×10^{-10}	-0.0154609127906
		2.4	-0.0161331264341	4.466×10^{-10}	-0.0161331263904
400	7125.34	2.0	-0.0339267631568	9.338×10^{-10}	-0.0339267630590
		2.1	-0.0356231013144	9.804×10^{-10}	-0.0356231012120
		2.2	-0.0373194394724	1.0270×10^{-9}	-0.0373194393650
		2.3	-0.0390157776302	1.0738×10^{-9}	-0.0390157775176
		2.4	-0.0407121157880	1.1204×10^{-9}	-0.0407121156709

Table 1.0: Change of the radial distance of a satellite moving in a Yukawa gravitational field in a circular (e=0) orbit around the Earth for various masses and satellite radial distances

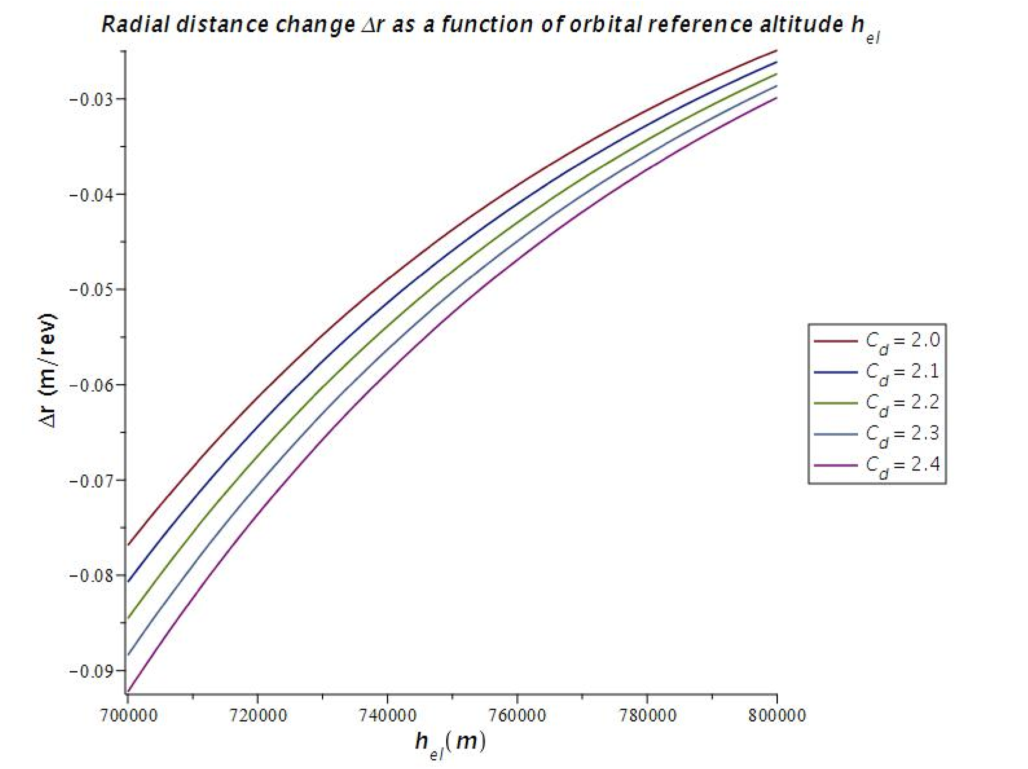

Figure 1.0: Radial distance change Δr as a function of orbital reference altitude h_{el} for a satellite of a surface area A_s= 3m² and mass m_s= 900kg in a circular orbit

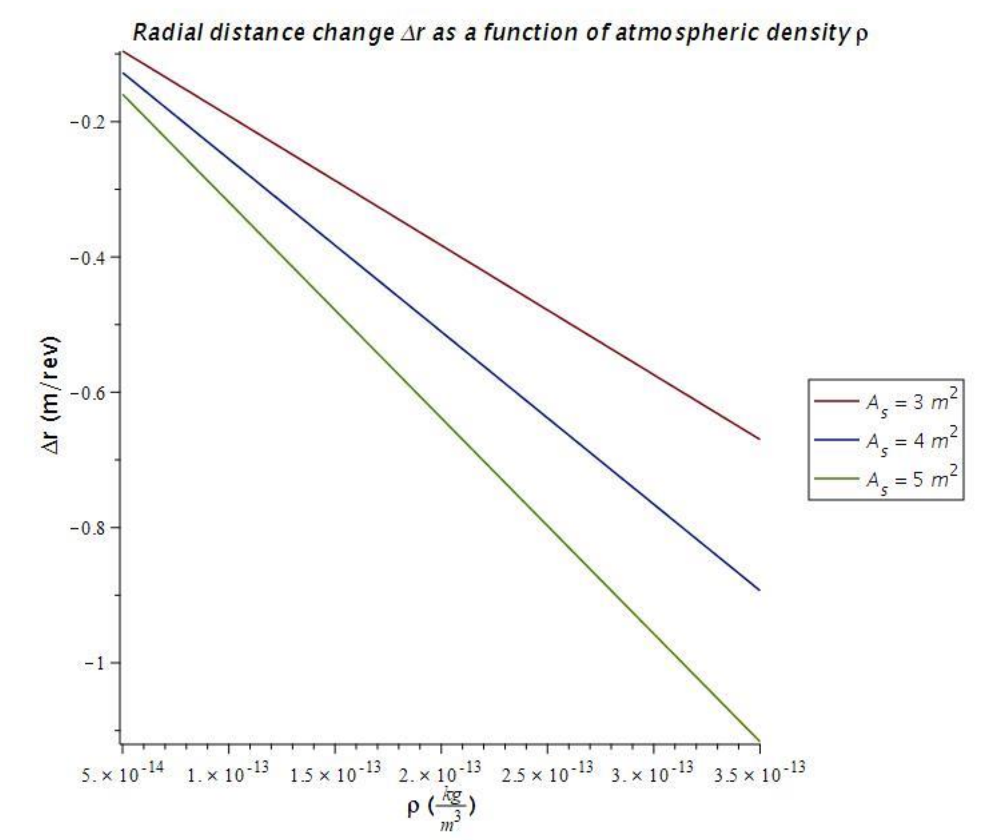

Figure 2.0: Radial distance change Δr as a function of atmospheric density for a satellite of mass m_s= 1000kg in a circular orbit

Conclusion

Using a correction to the typical Newtonian potential known as Yukawa potential, two systems with eccentricity equal to zero (circular orbits) are analyzed. For circular orbits, the Yukawa potential ranged in the order of magnitude of approximately 10^{-9} to 10^{-10} for realistic systems (as seen in Table 1.0). As radial distance between the two orbiting bodies approaches the range of interaction (λ) the difference between the Yukawa and Newtonian potential decreases; but this system is unrealistic for the case of satellite dynamics. Therefore, Yukawa potential on a satellite's motion in a circular orbit under the influence of atmospheric drag in a realistic system is trivial when compared to Newtonian potential. Further examination of Yukawa potential can be analyzed using bodies with greater mass such as planets and stars.

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